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Response of nonlinear systems in probability domain using stochastic averaging

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Abstract

Determination of probability density function (PDF) of the response for strongly nonlinear single-degree-of-freedom system subjected to both multiplicative and additive random excitations using stochastic averaging technique is faced with few difficulties. Firstly, the size of excitations should be small such that the response of the system converges weakly to a Markov process. Secondly, the excitations should preferably be broad banded so that the analytical results are reasonably accurate and finally, the nonlinear functions should be integrable if closed-formed expressions for the result are desired. The above issues are examined in the paper with a view to show (a) limiting values of the size parameter of excitations for which stochastic averaging technique can be applied to obtain reasonable estimates of PDF and mean square value of response, (b) the effect of the nature of the frequency contents of excitation on the response and (c) the use of the stochastic averaging method employing generalized harmonic functions for nonintegrable functions representing the dynamic system. It is shown that the stochastic averaging procedure provides results which compare very well with those obtained from simulation analysis not only for wide band excitations, but also for narrow band excitations for a wide range of the size parameter of excitation. Further, the procedure can be easily implemented for highly irregular functions by employing a numerical scheme with FFT.

1. Introduction

Response of nonlinear dynamic system subjected to stationary random excitations has been extensively studied using frequency-domain, time-domain and probability-domain techniques. Classical time-domain analysis is straightforward and relies on the simulation of excitation in the form of a time history of excitation of specified duration. The response of the system is then obtained for the specified time history of excitation, converting the problem to a deterministic one. The system nonlinearities are tackled by iterative procedure at each time step in time-marching integration scheme. Recently, stochastic numerical integration schemes have been developed for obtaining the nonlinear response of dynamic systems (Ray and Dash [1]). Although system nonlinearities are best handled in time-domain analysis, it has the disadvantage that it takes much computational time and its convergence to steady-state solution depends on the choice of initial condition.

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Most of the frequency-domain techniques used for response analysis for nonlinear dynamic systems adopt stochastic linearization procedure for linearizing the problem (Cai et al. [2]; Malhotra and Penzien [3]; Rajagopalan and Eatocktaylor [4]; Roberts and Spanos [5]). The iterative frequency-domain approach using Newton raphson technique has also been used to obtain the nonlinear response of the system (Datta and Jain [6]). The frequency-domain approach is disadvantaged by the fact that it does not work well for strongly nonlinear system and also, for near resonating condition. Both time-domain and frequency-domain techniques can be easily extended to MDOF system and for both narrow and wide band excitations. Nonlinear response in probability domain essentially uses Itô differential equation and the corresponding FPK equation. For nonlinear systems, use of FPK equation is particularly favoured by many investigators for Gaussian white noise excitation (Haung et al. [7]; Blankenship and Papanicolaou [8]; Soize [9]; Fuller [10]). Stochastic averaging technique has been extensively used for obtaining the response of strongly nonlinear system using FPK equation. Later, stochastic averaging has been extended to the case of wide-band random excitation (Cai and Lin [11]; Zhu et al. [17]), and combined harmonic and Gaussian white noise excitation (Huang and Zhu [12]). The extension of probability domain technique to MDOF system under general type of excitation (both narrow and wide band) is met with several complexities. Only for a class of problem and excitations, probability domain technique can be successfully implemented for MDOF systems.

Stochastic averaging method, which is extensively used for probability domain analysis was initially proposed by Landu and Stratnovich [13], and Khasminiskii [14], later modified by Zhu [15], and Zhu and Lin [16], for SDOF system excited by Gaussian white noise. For wide band random excitation the method has been extended by several researchers (Haung et al. [7]; Zhu et al. [17]). The application of the method for MDOF system have been accomplished for response analysis of quasi-Hamiltonian systems subjected to Gaussian white noise (Zhu and Yang [18]; Haung and Zhu [19]; Zhu and Haung [20] and [21]; Zhu et al. [22]; Huang et al. [23]; Cai and Lin [24]). For wide band excitation, the use of stochastic averaging technique to find response of strongly nonlinear MDOF system is not widely reported.

Although stochastic averaging method has been used for a variety of cases some of the issues that were not addressed in the papers are (i) validity of the method with respect to the size of excitation and damping related functions, (ii) applicability of the procedure for narrow band excitation and (iii) use of the method for problems having nonlinearities, which do not permit closed-form expressions to be obtained purely analytically. In the present paper, the above issues are investigated by solving two illustrative examples using a numerically based method developed for stochastic averaging procedure.

2. Theory

Consider free vibration of a nonlinear SDOF system without damping. The equation of motion is

$$\ddot{x} + f(x) = 0. \tag{1}$$

The total energy E of the system is given by

$$E = \frac{1}{2}\dot{x}^2 + V(x), \quad V(x) = \int_0^x f(z)\,\mathrm{d}z. \tag{2a,b}$$

System to be stable, $f(x)$ and $V(x)$ will be such that Eq. (1) has periodic solutions surrounding the origin in the phase plane $(x, \dot{x})$ with the origin as an equilibrium point. The periodic solution of Eq. (1) can be conveniently written in the form

$$x(t) = a\cos\varphi(t) + b, \quad \varphi(t) = \psi(t) + \theta. \tag{3a,b}$$

In which a, b and θ are constants. It can be shown that [17]

$$\dot{x}(t) = -a\beta(a,\varphi)\sin\varphi(t), \quad \beta(a,\varphi) = \sqrt{\frac{2[V(a+b) - V(a\cos\varphi + b)]}{a^2\sin^2\varphi}}, \tag{4a,b}$$

$\cos\varphi(t)$ and $\sin\varphi(t)$ are called generalized harmonic functions having an instantaneous frequency of oscillation as $\beta(a,\varphi)$. For time averaging, $\varphi(t)$ may be replaced by

$$\varphi(t) = \omega(a)t + \theta. \tag{5}$$

In which $\omega(a)$ is the averaging of the function $\beta(a,\varphi)$ over a period. With the above concept of periodic motion of the free vibration of undamped nonlinear system about an equilibrium point with an averaged frequency, the stochastic averaging procedure using generalized harmonic function for obtaining the probability of response of a nonlinear SDOF system is briefly given below. The details are available in Ref. [17].

2.1. Response of the system using stochastic averaging procedure

For a nonlinear SDOF system subjected to both additive and multiplicative excitations, the equation of motion can be expressed in the form

$$\ddot{x} + f(x) = \varepsilon g(x,\dot{x}) + \varepsilon^{1/2}\sum_{k=1}^{m} g_k(x,\dot{x})\eta_k(t). \tag{6}$$

In which ε is a very small quantity denoting $g(x,\dot{x})$ and $g_k(x,\dot{x})\eta_k(t)$ to be small; $f(x)$ is the nonlinear function denoting the restoring action. For small values of ε, the motion of the system will be nearly periodic and the response can be expressed by similar expressions as Eqs. (3a) and (4a) assuming a, φ, ψ and θ to be random processes. The solution of Eq. (6) in the form of Eqs. (3a) and (4a) with random parameters can be regarded as a set of random Van-Der-Pol transformation from x, $\dot{x}$ to a and φ. Assuming $x(t)$, $a(t)$, $\varphi(t)$ as random processes and extending Eqs. (3a) and (4a) for random process, $x(t)$ and $\dot{x}(t)$ can be written as

$$x(t) = a(t)\cos\varphi(t) + b, \tag{7}$$

$$\varphi(t) = \psi(t) + \theta(t), \tag{8}$$

$$\dot{x}(t) = -a(t)\beta(a,\varphi)\sin\varphi(t) \tag{9}$$

in which $\beta(a,\varphi)$ is given by Eq. (4b). Using the generalized harmonic functions Eqs. (7)–(9), it can be shown that Eq. (6) can be written as two first-order differential equations in the transformed domain as

$$\dot{a} = \varepsilon q_1(a,\varphi) + \varepsilon^{1/2}\sum_{k=1}^{m}\sigma_{1k}(a,\varphi)\eta_k, \tag{10a}$$

$$\dot{\varphi} = \varepsilon q_2(a,\varphi) + \varepsilon^{1/2}\sum_{k=1}^{m}\sigma_{2k}(a,\varphi)\eta_k \tag{10b}$$

in which

$$q_1(a,\varphi) = -a\bar{f}g(a,\varphi)\beta(a,\varphi)\sin\varphi, \quad q_2(a,\varphi) = q_1(\cos\varphi + h)/a\sin\varphi, \tag{11a,b}$$

$$\sigma_{1k}(a,\varphi) = -a\bar{f}g_k(a,\varphi)\beta(a,\varphi)\sin\varphi, \quad \sigma_{2k}(a,\varphi) = \sigma_{1k}(\cos\varphi + h)/a\sin\varphi \tag{11c,d}$$

in which $\bar{f} = 1/f(a+b)/(1+h)$; $g(a,\varphi) = g[(a\cos\varphi + b), -a\beta(a,\varphi)\sin\varphi]$; $g_k(a,\varphi) = g_k[(a\cos\varphi + b), -a\beta(a,\varphi)\sin\varphi]$; $\beta(a,\varphi)$ is defined earlier.

Since functions $q_1(a,\varphi)$, $\sigma_{1k}(a,\varphi)$ etc are periodic, they can be Fourier synthesized using FFT for different assumed values of 'a'. Thus, σ^r_{1kn}, σ_{2ln}, q_{10}, etc. are obtained from the real and imaginary parts of the FFT of $q_1(a,\varphi)$, $\sigma_{1k}(a,\varphi)$, etc. They are later used to obtain the drift and diffusion coefficients of Itô equation.

Effective band width of random excitation $\eta(t)$ in Eq. (10a) and (10b) depends on the value of ε. As $\varepsilon \to 0$, effective band width tends to infinity and process $a(t)$ converges weakly to a diffusive Markov process. The Itô equation of the limiting diffusion process represented by Eq. (10) is of the standard form [25]

$$\mathrm{d}a = u(a)\,\mathrm{d}t + \sigma(a)\,\mathrm{d}B(t) \tag{12}$$

in which $u(a)$ and $\sigma(a)$ are the averaged drift and diffusion coefficients obtained using stochastic averaging procedure [25]. The expressions for $u(a)$and $\sigma^2(a)$ using frequency components of excitation and functions q_1, σ_{1k}, etc. are given by [17]

$$u(a) = q_{10} + \sum_{k=1}^{m}\sum_{l=1}^{m} \pi\sigma'_{1ko}\sigma_{1lo}s_{kl}(o) + \sum_{k=1}^{m}\sum_{l=1}^{m}\left[\frac{\pi}{2}\sum_{n=1}^{\infty}\{(\sigma^r_{1kn})'\sigma^c_{1ln} + (\sigma^i_{1kn})'\sigma^i_{1ln} + n(\sigma^i_{1kn}\sigma^r_{2ln} - \sigma^r_{1kn}\sigma^i_{2ln})\}S_{kl}(n\omega(a))\right], \tag{13}$$

$$\sigma^2(a) = \sum_{k=1}^{m}\sum_{l=1}^{m} 2\pi\sigma_{1ko}\sigma_{1lo}s_{kl}(o) + \sum_{k=1}^{m}\sum_{l=1}^{m}\left[\pi\sum_{n=1}^{\infty}(\sigma^r_{1kn}\sigma^r_{1ln} + \sigma^i_{1kn}\sigma^i_{1ln})S_{kl}(n\omega(a))\right] \tag{14}$$

in which S_{kl} is the cross power spectral density function between the process η_k and η_l; $(\sigma_{1l}{}^r)'$, etc. is the derivative of $\sigma_{1l}{}^r$ with respect to 'a'.

The averaged FPK equation associated with Ito's equation Eq. (12) is of the standard form

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial a}[u(a)p] + \frac{1}{2}\frac{\partial^2}{\partial a^2}\left[\sigma^2(a)p\right] \tag{15}$$

in which $p = p(a,t|a_0,t_0)$ is the transition probability density of displacement amplitude.

The initial condition of FPK equation is

$$p = \delta(a - a_0), \quad t = 0. \tag{16}$$

The two boundaries of the FPK equation are $a = 0$ and ∞, if nonlinear restoring action exists. $a = 0$ is a regular boundary for nonzero external excitation, while $a = \infty$ is a singular boundary. For non zero external excitation, the stationary solution of FPK equation under the assumption of zero probability flow at the two boundaries is of the form [25]

$$p(a) = \frac{c}{\sigma^2(a)}\exp\left[\int_0^a \frac{2u(s)}{\sigma^2(s)}\,\mathrm{d}s\right], \tag{17}$$

where c is the normalization constant. For an assumed value of 'a', the numerical value of $p(a)$ can be obtained from Eq. (17) using the computed values of $u(a)$ and $\sigma^2(a)$ from Eqs. (13) and (14). Note that the values of variables of Eqs. (13) and (14) are obtained from the results of FFT. Once $p(a)$ is obtained, $p(x,\dot{x})$ and $p(x)$ are determined using the following relationships:

$$p(x,\dot{x}) = \left.\frac{p(E)}{T(E)}\right|_{E=\dot{x}(2/2)+v(X)}, \quad p(E) = \frac{p(a)}{f(a+b)}, \tag{18a,b}$$

where $T(E)$ is obtained from $T(a) = 2\pi/\omega(a)$ (with $\omega(a)$ as the average of the function $\beta(a,\varphi)$ taken over a period) by replacing 'a' by E, in which E is given by $E = V(a+b)$. $p(x)$ is obtained as

$$p(x) = \int_{-\infty}^{\infty} p(x,\dot{x})\,\mathrm{d}\dot{x}. \tag{19}$$

3. Application to Duffing Van-Der-Pol oscillator

The same Duffing-Van-Der-Pol oscillator with both additive and multiplicative excitations considered by Zhu [17], is taken as the first example. The equation of motion of the system is of the form

$$\ddot{x} + (-\beta_1 + \beta_2 x^2)\dot{x} + \omega_s^2 x + x^3 = xF_1(t) + F_2(t), \tag{20}$$

where ω_s, β_1, β_2 are constant, $F_1(t)$ and $F_2(t)$ are stationary and ergodic process with zero mean and rational power spectral densities. The oscillator given by Eq. (20) when cast in the form of Eq. (6), provides

$$f(x) = \omega_s^2 x + x^3, \quad g(x,\dot{x}) = -(-\beta_1 + \beta_2 x^2)\dot{x}, \quad g_1(x,\dot{x})\eta_1 = xF_1(t), \quad g_2(x,\dot{x})\eta_2 = F_2(t). \tag{21a–d}$$

The expressions for $V(x)$, E, $\beta(a,\varphi)$, $q_1(a,\varphi)$, etc. remain same as that given in Ref. [17].

Instead of following the closed-form analytical approach for the problem as carried out by Zhu [17], the following numerical computational scheme is used to develop a computer program in MATLAB:

1. Assume a set of values of 'a' from zero to some desired maximum value at small interval (which dictates the accuracy).
2. Choose φ values at equal intervals between 0 and 2π (number of values may preferably be 2^n).
3. For each value of 'a', obtain FFT of q_1, σ_{11}, σ_{12}, σ_{21}, and σ_{22} and obtain $T(a)$, $\omega(a)$, q_{10}; $\sigma^r_{11n}, \sigma^r_{12n}, \sigma^r_{21n}$ and $\sigma^r_{22n}, \sigma^r_{11n}, \sigma^r_{12n}$ and σ^r_{22n} for $n = 1$ to m (value of m to be selected depends upon the cut off frequency of the PSDF of excitation).
4. Once a set of values σ^r_{11n}, etc. are obtained, $(\sigma^r_{11n})'$ is determined by three points numerical differentiation.
5. $u(a)$ and $\sigma^2(a)$ values are obtain from Eqs. (13) and (14).
6. $p(a)$ is determined from Eq. (17) for each value of 'a'; the integration in Eq. (17) is performed numerically.
7. For each value of 'a', E is computed and $p(E)$ is obtained from Eq. (18b).
8. For different combinations of x and $\dot{x}$, E is obtained and $p(x, \dot{x})$ is computed from Eq. (18a).
9. $p(x)$ is obtained by numerical integration of $p(x, \dot{x})$.

3.1. Simulation analysis

Time histories of $F_1 = F_2$ are simulated from given PSDFs of $F_1(t)$ and $F_2(t)$ using the standard simulation procedure. In order to obtain sufficiently smooth PDF of the response, long time history of $F(t)$ is simulated with Δt as small as required. The value of Δt depends on cut of frequency of PSDF, time period and nonlinearity of the system. For the simulated time histories of $F_1(t)$, $F_2(t)$ the time history of $x(t)$ is obtained using Newmark's β method with iteration performed at each time step to consider the nonlinearities. For this purpose, all nonlinear terms of Eq. (6) are taken to the right-hand side of the equation and treated as known, equal to those in the previous time step, in the beginning of the iteration. In subsequent iterations, they are updated till convergence is achieved. From the time history of $x(t)$, $p(x)$ is obtained using MATLAB.

The time history of $a(t)$ is obtained from that of $x(t)$ using the following transformation:

$$V(a) = \int_0^a f(x)\,\mathrm{d}x = \frac{1}{2}\omega_s^2 a^2 + \frac{1}{4}a^4. \tag{22}$$

The total energy E is given by

$$E = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\omega_s^2 x^2 + \tfrac{1}{4}x^4 = V(a). \tag{23}$$

Equating Eqs. (22) and (23), a quadratic equation in a^2 is obtained

$$\tfrac{1}{4}a^4 + \tfrac{1}{2}\omega_s^2 a^2 = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\omega_s^2 x^2 + \tfrac{1}{4}x^4. \tag{24}$$

As time histories of $x(t)$ and $\dot{x}(t)$ are known, the time history of $a(t)$ can be obtained by solving Eq. (24). Thus, for each time history of $x(t)$, a corresponding time history of $a(t)$ is obtained. $p(a)$ is obtained from the time history of $a(t)$ in the same way as $p(x)$ is obtained.

3.2. Discussion of results

Results of the Van-Der-Pol oscillator problem, Eq. (20), obtained by numerical procedure are compared with the closed-form solution provided by Zhu [17] and a detail parametric study is carried out to show the applicability of stochastic averaging procedure. For the purpose of analysis, the values of β_1 and β_2 are taken as 0.01 and 0.02, respectively, and ω_s is taken as unity.

Since Eqs. (6) and (20) are divided by mass (m) all through, the excitations η_k or $f_1(t)$, etc. are normalized excitations (with respect to mass). For simplicity of calculation, 'm' may be taken as unity. Then $F_1(t)$ and $F_2(t)$ in Eq. (20) may be taken as the actual excitations.

The power spectral density function of $F_1(t) = F_2(t) = F(t)$ is taken to be of the form

$$s_f(\omega) = \frac{D}{\pi(\omega^2 - \omega_f^2)^2 + 4\pi\xi_f^2\omega_f^2\omega^2}. \tag{25}$$

By changing the values of ω_f and ξ_f, the shape of the PSDF can be changed from narrow to wide band. By assigning suitable value to D, normalized excitation (with respect to mass) in Eqs. (6) and (20) can be made very small i.e. ($\varepsilon \to 0$). The measure of ε is given by the parameter ρ defined by $\rho = \sigma_f/m$, in which σ_f^2 is the area under curve of PSDF of excitation and 'm' is the mass of the system. As will be shown later, the measure of ε is difficult to define since it also depends upon band width of the frequency of excitation and the ratio of predominant frequency of excitation to the system frequency (initial). Since $m = 1$ in the present problem, σ_f^2 is the area under the curve of the PSDF of $F(t)$. The PSDFs of both multiplicative $F_1(t)$ and additive $F_2(t)$ excitations, taken as the same, is shown in Fig. 1 for $D = 0.2$; $\omega_f = 5$; $\xi_f = 0.5$. The probability density functions (PDFs) $p(a)$ and $p(x)$ as obtained from the proposed method and simulation results are shown in Figs. 4 and 5, respectively, for the values of $\rho = 0.028$ and 0.252. For $D = 0.2$, ρ is calculated as 0.028 while

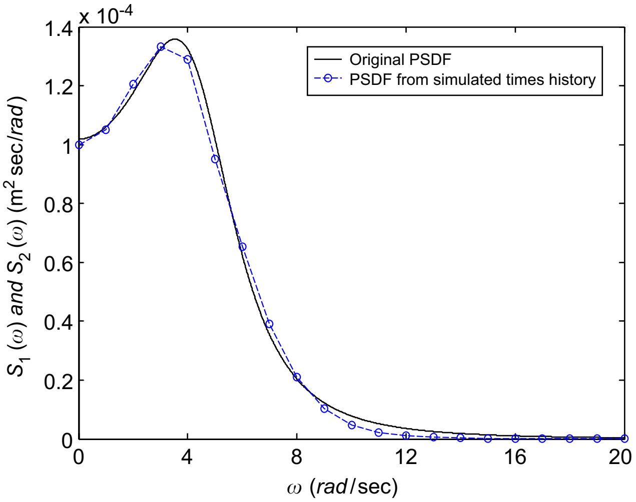

Fig. 1. PSDF of excitations ($F_1 = F_2 = F$) defined by Eq. (52); $\omega_f = 5$; $D = 0.2$; $\xi_f = 0.5$ and PSDF obtained from simulated time history.

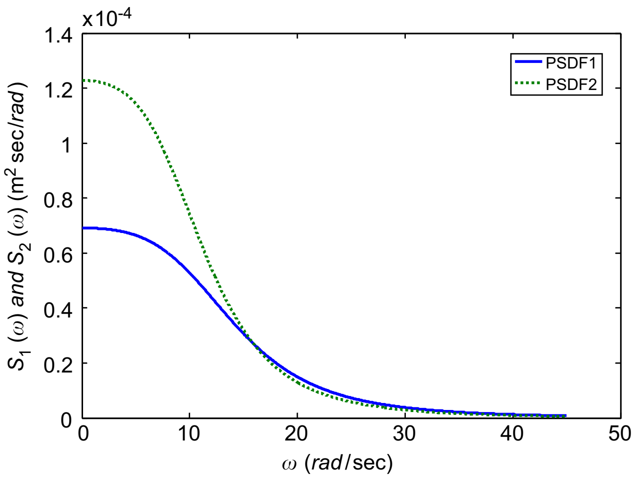

Fig. 2. PSDFs of broad band excitations; PSDF1 (for F_1) $\omega_f = 15$; $D = 11$; $\xi_f = 0.75$, PSDF2 (for F_2) $\omega_f = 12$; $D = 8$; $\xi_f = 0.75$.

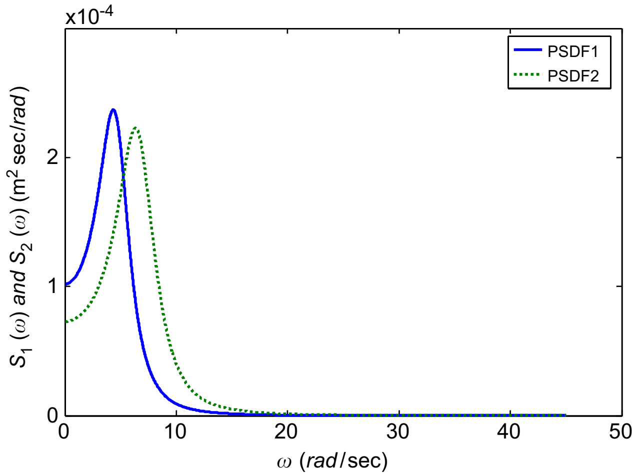

Fig. 3. PSDFs of narrow band excitations; PSDF1 (for F_1) $\omega_f = 5$; $D = 0.2$; $\xi_f = 0.35$, PSDF2 (for F_2) $\omega_f = 7$; $D = 0.55$; $\xi_f = 0.3$.

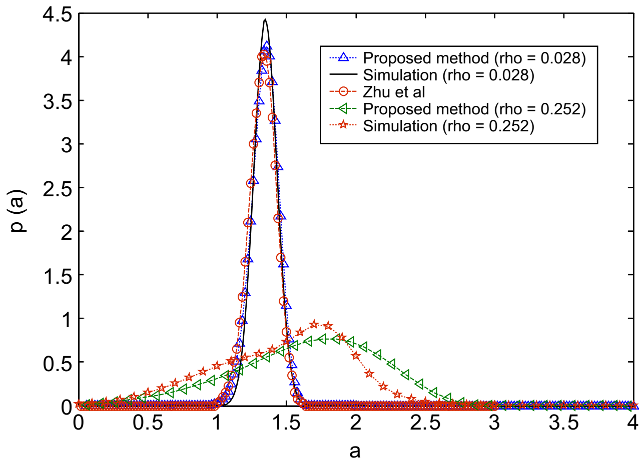

Fig. 4. PDFs of the displacement amplitude (a) for the Duffing oscillator ($\rho = 0.028$ and 0.252); excitation shown in Fig. 1.

'D' is increased to 16 for making $\rho = 0.252$. It is seen from the figures that for the small value of ρ i.e., $\rho = 0.028$, the agreement between the two results are good. Further, the results compare very well with those obtained by Zhu et al. [17]. For the higher value of $\rho = 0.252$, the simulation results deviate from those of the proposed method; the difference between the rms response obtained from the two is about 22% (Figs. 4 and 5).

In order to ensure that simulation results are correct, the smoothed PSDF of $F_1(=F_2)$ of Eq. (20) obtained from the simulated time history of excitation F_1, is compared with the parent PSDF given by Eq. (25) in Fig. 1. It is seen from the figure that the both agree extremely well. Note that the simulated PDF is obtained by averaging the results of four simulations; each simulated time history of response is of 50 min duration.

For studying the effect of the parameter ρ on the accuracy of the results obtained, it is varied over a range of values. The same Van-Der-Pol oscillator is analyzed with PSDFs of excitations as shown in Fig. 2. These PSDFs are relatively broad band. The value of ρ is varied by varying the value of 'D' as mentioned before. For effecting the change in ρ the values of 'D' for both excitations $F_1(t)$ and $F_2(t)$ in Fig. 2 are changed in the same proportion. For the purpose of discussing the results, ρ value for $F_2(t)$ ($F_1(t) < F_2(t)$) is used and change in 'D' is made such that it varies from 0.039 to 0.39. Note that 'D' for $F_1(t)$ is also changed in the same proportion.

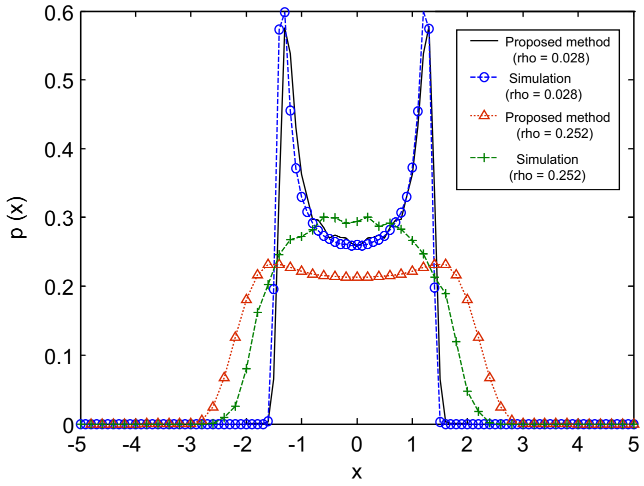

Fig. 5. PDFs of displacement (x) for Duffing oscillator ($\rho = 0.028$ and 0.252); excitation shown in Fig. 1.

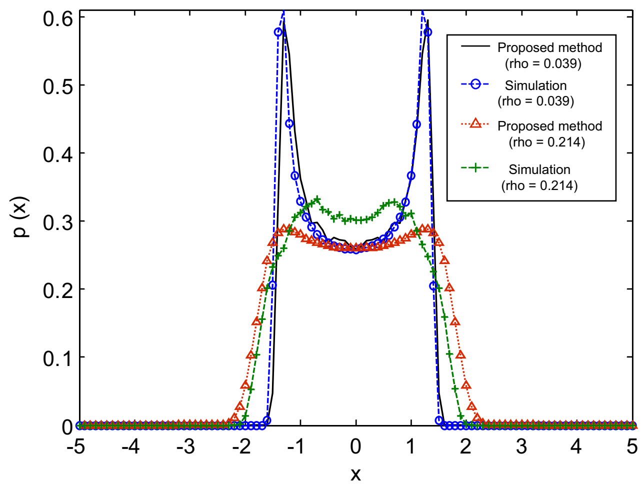

Fig. 6. PDFs of displacement (x) for Duffing oscillator ($\rho = 0.039$ and 0.214); excitations shown in Fig. 2.

The PDFs of the response, $p(x)$ as obtained from the proposed method and simulation are compared in Fig. 6 for $\rho = 0.039$ and 0.214. It can be seen from the figure that the two results compare extremely well for $\rho = 0.039$. Further, the two results agree reasonably well for $\rho = 0.214$; the difference between the rms response is about 12%. However, for $\rho = 0.392$, the simulation results deviate considerably from that obtained from the proposed method as shown in Fig. 7; the difference between the rms response is about 22%. For the same oscillator, when the excitations presented by the PSDF shown in Fig. 1 was used, the deviations between the theoretical and simulation results for $p(a)$ and $p(x)$ was significant for $\rho = 0.252$, as shown in Figs. 4 and 5. Thus, the range of ρ values for which the stochastic averaging procedure provides reasonably good results depend upon the nature of excitation. This is further demonstrated by obtaining the responses of the same oscillator for relatively narrow band excitations represented by PSDFs shown in Fig. 3. The plots of $p(x)$ are shown in Figs. 8 and 9. Here again, ρ represents the size parameter of $F_2(t)$. It is seen from Fig. 8 that for $\rho = 0.036$, the simulation and analytical results compare extremely well. For $\rho = 0.2$, the two results agree reasonably well; the difference between rms response is about 10%. For $\rho = 0.365$, considerable deviation between the two results is observed, as shown in Fig. 9; the difference between the rms responses is about 22%.

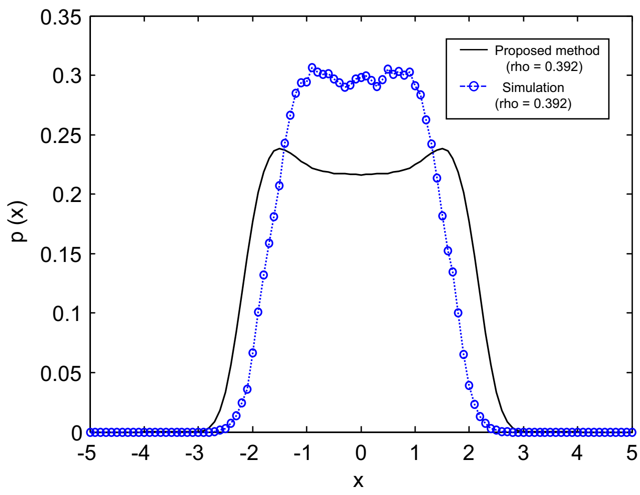

Fig. 7. PDFs of displacement (x) for Duffing oscillator($\rho = 0.392$); excitations shown in Fig. 2.

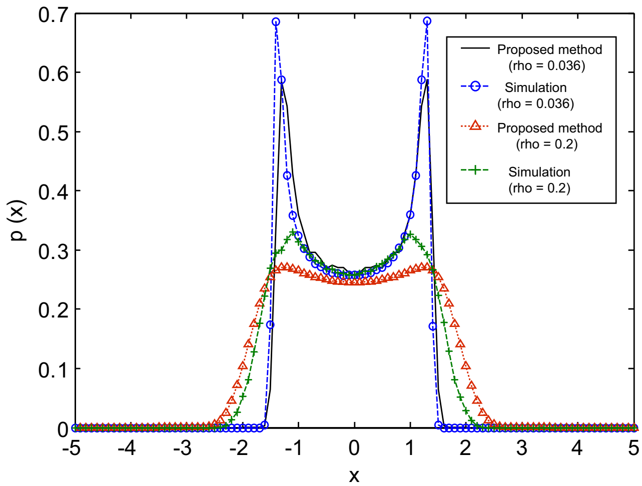

Fig. 8. PDFs of displacement (x) for Duffing oscillator ($\rho = 0.036$ and 0.2); excitations shown in Fig. 3.

Note that for narrow band excitation the predominant frequency of excitation as observed from the PSDF is 6.5. Thus, the ratio of the excitation frequency and the initial structural frequency, ω_s is 6.5. To check the applicability of method for near resonance condition (i.e. the ratio of initial structural frequency and excitation frequency is nearly unity), a case is considered in the next problem. A study of the stochastic averaging technique for near resonance condition was studied by Huang et al. for excitation which is a combination of white noise and harmonic excitation [26].

In most of the engineering applications, mean square or rms value of the response is of greater interest. For the above example problem, the percentage error in rms response compared to that obtained from simulation analysis is shown in Fig. 10 as a function of ρ for both narrow and wide band excitations. It is seen from the figure that rms response obtained by the proposed method is less than the simulated one and the difference remains within 15% over a wide range of value of the ρ i.e. $\rho = 0.03$ to 0.3 for wide band excitation, and $\rho = 0.03$ to 0.2 for narrow band excitation. This clearly shows that the stochastic averaging procedure can be applied for finding the rms response of nonlinear systems for a wide range of values of ρ (not necessarily for very small value only). The range depends upon the nature of excitations and the nonlinearities.

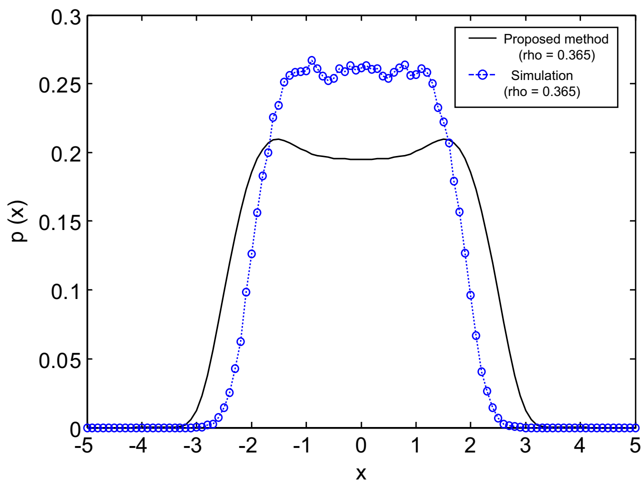

Fig. 9. PDFs of displacement (x) for Duffing oscillator ($\rho = 0.365$); excitations shown in Fig. 3.

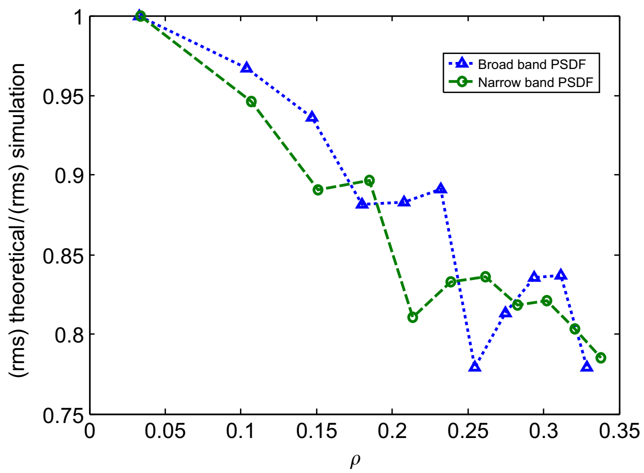

Fig. 10. Variation of the ratio of (the rms of x) theoretical to (the rms of x) simulation with ρ for Duffing oscillator.

4. Application to a nonlinear oscillator (with nonintegrable function)

The use of the same method and the effect of ρ on the response are investigated for a nonlinear oscillator whose nonlinearities can not be handled in a closed-form solution (second problem). The equation of motion of the oscillator is given by

$$\ddot{x} + (\beta_1 + \beta_2 x^2 + \beta_3 x^3)\dot{x}|\dot{x}| + \omega_s^2 x + \alpha x^3 = |\dot{x}|x\xi_1 + \xi_2. \tag{26}$$

When Eq. (26) is cast in the form of Eq. (6), it provides

$$f(x) = \omega_s^2 x + x^3; \quad g(x,\dot{x}) = -(\beta_1 + \beta_2 x^2 + \beta_3 x^3)\dot{x}|\dot{x}|; \quad g_1(x,\dot{x})\eta_1 = |\dot{x}|x\xi_1(t); \quad g_2(x,\dot{x})\eta_2 = 1\xi_2(t). \tag{27a–d}$$

The expressions for $V(x)$, E, $\beta(a,\varphi)$, $q_1(a,\varphi)$, etc. are obtained as

$$V(x) = \int_0^x f(u)\,\mathrm{d}u = \omega_s^2 \frac{x^2}{2} + \frac{x^4}{4}, \tag{28}$$

$$\beta(a,\varphi) = \left[\frac{2}{a^2 \sin^2\varphi}\left[\omega_s^2 \frac{a^2}{2} + \frac{a^4}{4} - \omega_s^2 \frac{a^2\cos^2\varphi}{2} - \frac{a^4\cos^4\varphi}{4}\right]\right]^{1/2}, \tag{29}$$

$$q_1(a,\varphi) = -\frac{a^2}{f(a)}(\beta_1 + \beta_2 a^2\cos^2\varphi + \beta_3 a^3\cos^3\varphi)(\beta(a,\varphi)\sin\varphi)^2|-a\beta(a,\varphi)\sin\varphi|, \tag{30}$$

$$q_2(a,\varphi) = -\frac{a}{f(a)}(\beta_1 + \beta_2 a^2\cos^2\varphi + \beta_3 a^3\cos^3\varphi)(\beta(a,\varphi))^2\sin\varphi\cos\varphi|-a\beta(a,\varphi)\sin\varphi|, \tag{31}$$

$$\sigma_{11}(a,\varphi) = -\frac{a^2}{f(a)}|-a\beta(a,\varphi)\sin\varphi|\beta(a,\varphi)\sin\varphi\cos\varphi, \tag{32}$$

$$\sigma_{12}(a,\varphi) = -\frac{a}{f(a)}\beta(a,\varphi)\sin\varphi, \tag{33}$$

$$\sigma_{21}(a,\varphi) = -\frac{a}{f(a)}|-a\beta(a,\varphi)\sin\varphi|\beta(a,\varphi)\cos^2\varphi, \tag{34}$$

$$\sigma_{22}(a,\varphi) = -\frac{1}{f(a)}\beta(a,\varphi)\cos\varphi \tag{35}$$

in which $f(a) = {\omega_s}^3 a + a^3$.

The values of the parameters in Eq. (26) are taken as $\beta_1 = 0.005$; $\beta_2 = 0.01$; $\beta_3 = 0.015$; $\alpha = 1$ and $\omega_s = 1$. The results are obtained for excitations ξ_1 and ξ_2 having PSDFs shown in Fig. 3 (narrow band) and Fig. 2 (broad band). Keeping the rms values of ξ_1 and ξ_2 as the same, the results are also obtained when they are represented by band limited white noise having a cut off frequency of 30 rad/s. The plots of $p(a)$ and $p(x)$ are shown in Figs. 11–14. In the figures, ρ denotes the size parameter for $f_2(t)$. It is seen that for $\rho = 0.036$, the agreement between the simulation and proposed method is extremely well for $p(a)$ as shown in Fig. 11 and for $p(x)$ as shown in Fig. 12 for narrow band excitation. Further, it is seen from the same figures that considerable deviation between the two results is observed for $\rho = 0.164$; the difference between rms responses is about 25%. In Fig. 13, $p(x)$ as obtained from the proposed method and the simulation are compared for broad band excitation. It is seen from the figure that for $\rho = 0.039$, the agreement between the two is extremely well. Considerable deviation between the two is observed when $\rho = 0.303$; the difference between rms responses is about 26%. In Fig. 14, the same comparison is shown for band limited white noise. It is seen that the value of

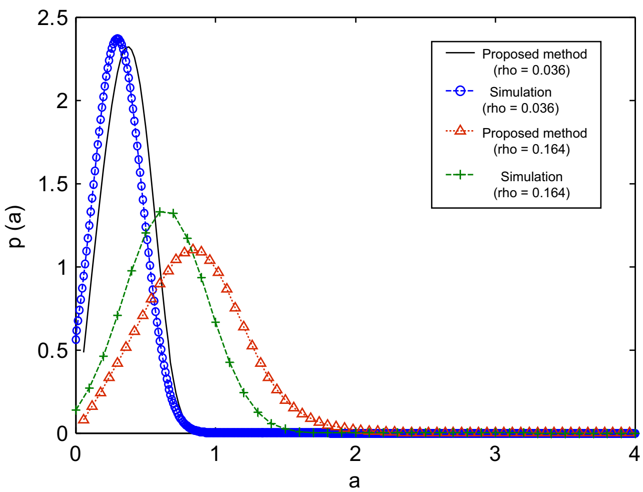

Fig. 11. PDFs for displacement amplitude (a) of the second problem ($\rho = 0.036$ and 0.164); narrow band excitations (Fig. 3).

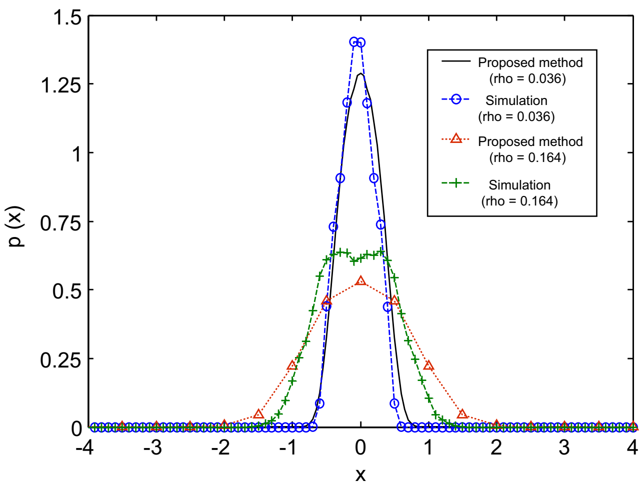

Fig. 12. PDFs of displacement (x) of the second problem ($\rho = 0.036$ and 0.164); narrow band excitations (Fig. 3).

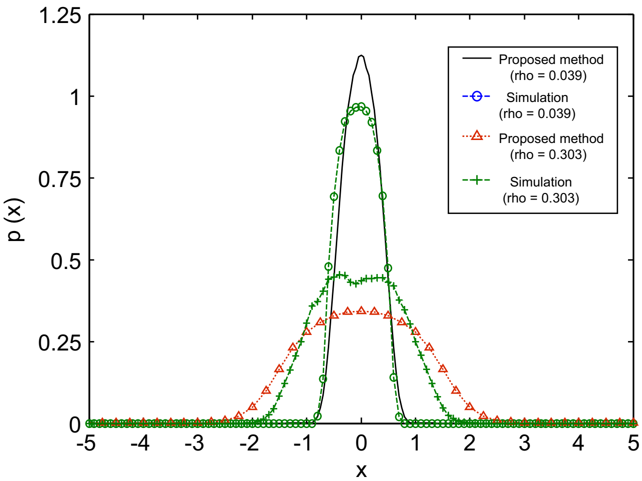

Fig. 13. PDFs of displacement (x) of the second problem ($\rho = 0.039$ and 0.303); broad band excitations (Fig. 2).

ρ for which deviation between the two results become significant is 0.392; the difference between the two rms responses is about 25%.

For investigating the applicability of method for near resonance condition, for narrow band excitation, the above problem is solved with $\omega_s = 6.5$ rad/s which is equal to the frequency where the peak of PSDF occurs. All other parameters remain the same for the problem. Results are plotted in Figs. 15 and 16. It is seen that for small value of $\rho = 0.036$, agreement between the results of simulation and proposed method is extremely well. Considerable deviation between the two results is observed at $\rho = 0.258$; the difference between the rms responses is about 25%. Thus, for near resonance condition under narrow band excitation, the stochastic averaging method provides better results as compared to non resonating condition (i.e. for $\omega_s = 1$) for which the same error is observed for $\rho = 0.164$. The reason for this is probably due to the effective strengths of excitations for the two cases. For the same PSDF of excitation shown in Fig. 3, the response at near resonance is predominantly governed by a strip of excitation centered around $\omega_s = 6.5$ rad/s. The area of the strip representing the effective strength of excitation may be smaller than that for the case of $\omega_s = 1$ rad/s.

The percentage error in rms response compared to that obtained from simulation analysis is shown in Fig. 17 as a function of ρ for the three types of excitations. It is seen from the figure that an error of about 15% in rms response is obtained for ρ values of 0.125, 0.151, 0.240 for narrow band, broad band and band limited

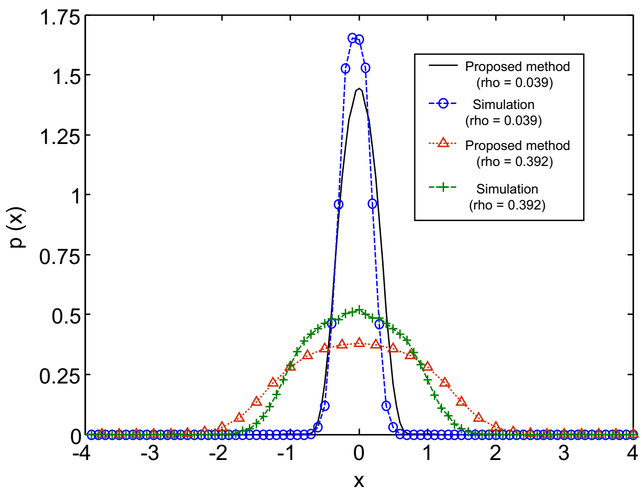

Fig. 14. PDFs of displacement (x) of the second problem ($\rho = 0.039$ and 0.392); band limited white noise excitation.

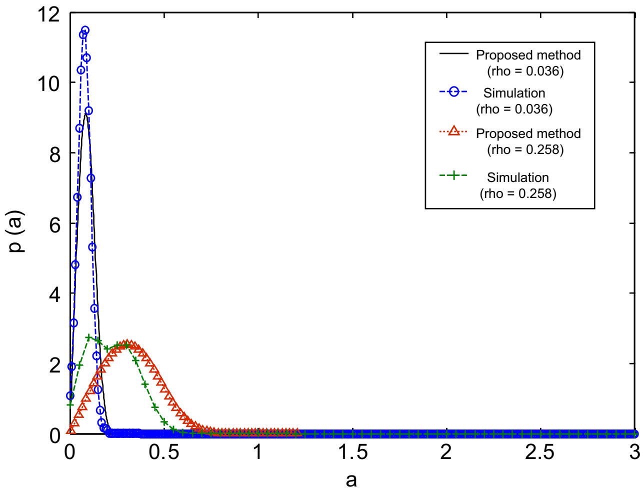

Fig. 15. PDFs for displacement amplitude (a) of the second problem with $\omega_s = 6.5$; ($\rho = 0.036$ and 0.258); narrow band excitations (Fig. 3).

white noise respectively (with $\omega_s = 1$). For narrow band excitation with $\omega_s = 6.5$, the error remains within 15% for $\rho \leqslant 0.152$. Thus, for this range of acceptable error, the performance of stochastic averaging technique is found better for near resonance condition for narrow band excitation for this particular problem.

5. Conclusions

A numerical approach using FFT for obtaining the response of nonlinear SDOF system in probability domain is presented for both multiplicative and additive stochastic excitations. The approach is based on the method originally proposed by Zhu et al. [17]. Using this approach a number of issues regarding the applicability of the stochastic averaging procedure using generalized harmonic function are investigated. Two numerical examples are considered for this purpose. The first example considers the usual Duffing oscillator subjected to both narrow and wide band excitations. The second example considers nonlinearities which are

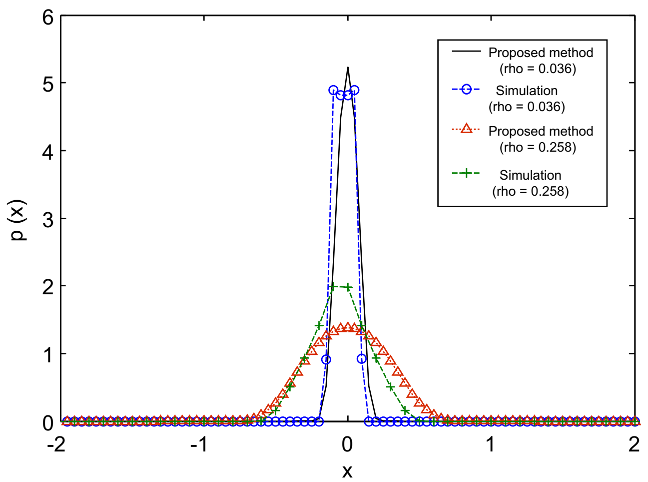

Fig. 16. PDFs of displacement (x) of the second problem with $\omega_s = 6.5$; ($\rho = 0.036$ and 0.258); narrow band excitations (Fig. 3).

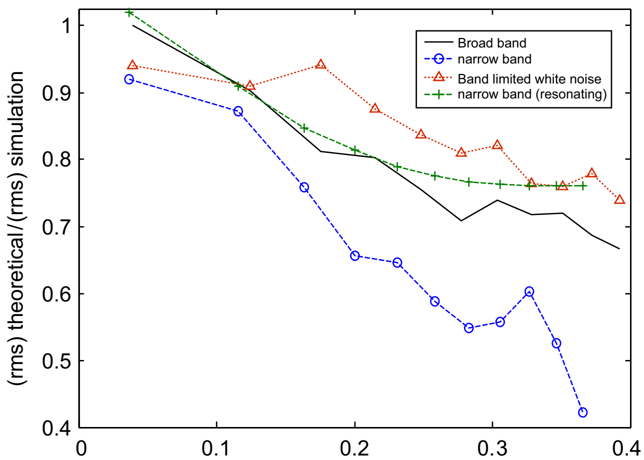

Fig. 17. Variation of the ratio of (the rms of x) theoretical to (the rms of x) simulation with ρ for the second problem.

not amenable to closed-form integration and is subjected to excitations ranging from narrow to band limited white noise excitation. The results of the study show that:

I. For small normalized excitation (normalized with respect to the mass of system), the responses obtained from the proposed numerical method compare well with those obtained from simulation for both narrow and wide band excitations.
II. For normalized excitations, which are not very small, the results obtained from the simulation analysis do not significantly differ from those obtained from the proposed method; the difference depends upon the nature of excitation. Thus, the applicability of stochastic averaging method is not limited to only very small excitations.
III. For both broad and narrow band excitations, the stochastic averaging technique provides reasonable estimates of rms response (within 15% error) for a wide range of the normalized excitation parameter (ρ) denoting the size of excitation.
IV. For near resonance condition, the performance of stochastic averaging is found to be better than nonresonance condition for narrow band excitation.
V. If the excitation can be modeled as a band limited white noise, good estimate of response can be obtained by the proposed method even for a sufficiently large value of ρ.

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